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Original Article

# Generalized BP-Recurrent Finsler Geometry with Applications to Intelligent Robotic Path Planning and Anisotropic Biomedical Systems

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## ABSTRACT

This paper presents an advanced geometric study of generalized BP-recurrent Finsler spaces $\mathcal{GBP}\text{-}RF_n$, where Cartan's second curvature tensor $P^i_{jkh}$ satisfies a first-order recurrence relation involving non-vanishing covariant vector fields. Necessary and sufficient conditions are derived for the non-vanishing of key curvature-related tensors, including the (hv)-torsion tensor, P-Ricci tensor, and deviation tensors.

The theoretical framework of $\mathcal{GBP}\text{-}RF_n$ spaces offers a robust mathematical model for analyzing physical and kinematic systems operating in anisotropic and non-Euclidean environments. Such properties make them well-suited for modern applications in intelligent robotics, where environmental inhomogeneity and directional sensitivity are crucial.

Specifically, the geometric structure of $\mathcal{GBP}\text{-}RF_n$ can be leveraged to enhance robot motion planning in complex environments, improve adaptive control algorithms, and optimize sensor-based navigation systems. By analyzing the recurrence relations of higher-order curvature tensors under Berwald differentiation, the paper provides a deep insight into how curvature and torsion influence feasible paths and motion dynamics.

Furthermore, the framework models targeted drug delivery in anisotropic biomedical systems using Finsler metrics, enhancing energy efficiency and trajectory stability of smart micro-navigational medical agents through generalized BP-recurrent geometry.

This work establishes a novel interdisciplinary bridge between higher-order differential geometry and real-world robotic systems, paving the way for advanced modeling of autonomous navigation, UAVs, and intelligent transportation in dynamically curved environments.

**Keywords**: Finsler Space, Cartan's Second Curvature Tensor $P^i_{jkh}$ , Generalized $\mathcal{BP}$-recurrent Space, Cartan's Fourth Curvature Tensor $R^i_{jkh}$ , Intelligent Robotic Systems, Adaptive Control.

## INTRODUCTION

Recent advancements in autonomous robotic systems have emphasized the need for geometric frameworks that can model environments where physical properties vary with both position and direction. Finsler geometry, particularly its generalized recurrent forms such as the GBP-$RF_n$ space studied here, provides a natural and rigorous mathematical tool for this purpose. By analyzing the recurrence of curvature tensors and their derivatives, this study lays the groundwork for employing such structures in real-world robotic motion planning, especially in anisotropic or curved environments where traditional Euclidean or Riemannian models fall short.

Finsler geometry has witnessed remarkable developments in recent years, particularly in the study of generalized recurrent structures and their associated curvature tensors. Several authors have extended the classical framework by introducing higher-order generalizations of recurrent spaces, focusing on the role of Berwald and Cartan connections in defining and analyzing special curvature tensors [1], [2], [3] and [4]. In particular, the Berwald Weyl curvature and its decompositions have been investigated as central objects in understanding the projective properties of Finsler spaces [3], while Schur-type lemmas and new invariants have been established to characterize the behavior of such spaces [1], [2].

Further contributions have explored the interrelations between Weyl's projective curvature tensor, R-projective tensors, and other curvature measures in both spacetime and generalized recurrent settings [5], [6] and [7]. These studies have opened new directions for analyzing Q-curvature inheritance and the role of Lie derivatives in generalized recurrent Finsler geometries [11], [12]. Moreover, higher-order recurrent spaces such as the generalized BK-fifth recurrent Finsler space have been systematically studied, highlighting the interplay between tensorial derivatives and the structure of projective motions [9], [10], [14].

Recent investigations also indicate that the introduction of new projective invariants within generalized Finsler metrics provides deeper insights into the intrinsic geometry of non-Riemannian spaces [2], [7]. Such results not only enrich the theoretical understanding of curvature tensors but also establish connections to broader geometric frameworks, including the Finsler–Laplace–Beltrami operators that appear in applied settings such as shape analysis [16].

Beyond pure geometry, the applications of Finsler structures have expanded into robotics and intelligent systems, where anisotropic and non-Euclidean environments play a critical role in navigation and path optimization. Modern algorithms for mobile robot path planning, particularly those based on improved RRT and related randomized methods, have been successfully employed in complex environments [17–19]. These approaches align naturally with the Finslerian perspective, were anisotropic metrics model heterogeneous motion costs and constraints. Recent studies on gradient estimates under Finsler–Ricci flows [20] further suggest potential applications in dynamic and adaptive environments, bridging the gap between theoretical developments in Finsler geometry and practical implementations in intelligent robotics.

Furthermore, anisotropic geometric structures modeled by Finsler metrics play a pivotal role in describing directed processes in chemical physics and biomedical transport phenomena. By incorporating generalized BP-recurrent curvature tensor properties, this paper not only establishes novel path optimization algorithms for intelligent autonomous robots but also models directional navigation strategies for biomedical micro-robots operating within complex, non-isotropic fluid environments.

In this context, the present study aims to contribute to the ongoing dialogue between the theoretical foundations of generalized recurrent Finsler spaces and their emerging applications. By focusing on projective and concircular motions, as well as the roles of higher-order recurrent structures, we seek to establish new insights into the relationship between curvature tensors and geometric invariants that govern both abstract mathematical structures and real-world navigation problems.

Consider an n-dimensional Finsler space equipped with a metric function F that satisfies the necessary regularity conditions. Let $g_{ij}$ denote the components of the associated metric tensor, while $\Gamma^{*i}_{jk}$ and $G^i_{jk}$ represent the parameters of Cartan's and Berwald's connections, respectively.

These connection coefficients are symmetric in their lower indices.

The vectors $y_i$ and $y^i$ satisfy the following relations:

(1) a) $y_i = g_{ij}\, y^j$ and b) $y_i\, y^i = F^2$ .

The metric tensor $g_{ij}$ and its reciprocal tensor $g^{ij}$ are connected by the relation:

(2) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 , & if \quad i = k , \\ 0 , & if \quad i \neq k . \end{cases}$, where $\delta^k_i$ is the Kronecker delta.

The tensor

(3) $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2$

is referred to as the hv-torsion tensor.

The (v)hv-torsion tensor $C^h_{ik}$ and its associated (h)hv-torsion tensor $C_{ijk}$ satisfy the relations:

(4) a) $C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0$ , b) $y_i\, C^i_{jk} = 0$ , c) $C_{ijk}\, y^j = 0$ ,
d) $g_{rj}\, C^r_{ik} = C_{ijk}$ , e) $C^i_{jk}\, g^{jk} = C^i$ and f) $C_{ijk}\, g^{jk} = C_i$ .

The Berwald covariant derivative of an arbitrary tensor field $T^i_j$ with respect to $x^k$ is defined as:

(5) $\mathcal{B}_k\, T^i_j = \partial_k\, T^i_j - (\dot{\partial}_r\, T^i_j)\, G^r_k + T^r_j\, G^i_{rk} - T^i_r\, G^r_{jk}$ .

For the metric function F, the vector $y^i$, and the normalized vector $l^i$, Berwald's covariant derivative vanishes identically:

(6) a) $\mathcal{B}_k\, y^i = 0$ , b) $\mathcal{B}_k\, F = 0$ and c) $\mathcal{B}_k\, y_i = 0$ .

However, the Berwald covariant derivative of the metric tensor does not vanish, i.e., $\mathcal{B}_k\, g_{ij} \neq 0$, and is given by:

(7) $\mathcal{B}_k g_{ij} = -2\, C_{ijk|h}\, y^h = -2\, y^h\, \mathcal{B}_h\, C_{ijk}$ .

Moreover, the Berwald covariant differential operator with respect to $x^h$ commutes with the partial derivative with respect to $y^k$, namely:(8) $(\dot{\partial}_k\, \mathcal{B}_h - \mathcal{B}_h\, \dot{\partial}_k)\, T^i_j = T^r_j\, G^i_{khr} - T^i_r\, G^r_{khj}$ , where $T^i_j$ is any arbitrary tensor field.

The hv-curvature tensor $P^i_{jkh}$ and the v(hv)-torsion tensor $P^i_{kh}$ satisfy the following conditions:

(9) a) $P^i_{jkh}\, y^j = P^i_{kh}$ , b) $g_{ir}\, P^r_{jkh} = P_{ijkh}$ , c) $g_{rp}\, P^r_{kh} = P_{kph}$ ,
d) $P^i_{jki} = P_{jk}$ , e) $P^i_{ki} = P_k$ , f) $P^i_{kh}\, y^k = P^i_{kh}\, y^h = 0$ , and g) $P^i_i = P$ .

The hv-curvature tensor $P^i_{jkh}$ can equivalently be expressed as:

(10) a) $P^i_{jkh} = \Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jh|k}$ , with the identities:
b) $P^i_{jkh} = \dot{\partial}_h\, \Gamma^{*i}_{jk} + C^i_{jr}\, C^r_{kh|s}\, y^s - C^i_{jh|k}$
c) $P^i_{jkh} = C^i_{kh|j} - C_{jkh|r}\, g^{ir} + C^r_{jk}\, P^i_{rh} - P^r_{jh}\, C^i_{rk}$ , were

(1.11) a) $\Gamma^{*i}_{jkh}\, y^j = P^i_{kh}$ , b) $\Gamma^{*i}_{jkh}\, y^k = 0$ and g) $y_i\, \Gamma^{*i}_{kjh} = -P_{kjh}$ .

Finally, the projective curvature tensor $W^i_{jkh}$ (also known as Weyl's projective curvature tensor), the projective torsion tensor $W^i_{jk}$ (Weyl's torsion tensor), and the projective deviation tensor $W^i_j$ (Weyl's deviation tensor) are defined by

(12) $$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{n+1} H_{[hk]} + \frac{2\,y^i}{n+1}\dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{n^2-1}\left(n\, H_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}\right) - \frac{\delta^i_h}{n^2-1}\left(n\, H_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr}\right) ,$$

(13) $$W^i_{jk} = H^i_{jk} + \frac{y^i}{n+1} H_{[jk]} + 2\left\{\frac{\delta^i_{[j}}{n^2-1}\left(n\, H_{k]} - y^r H_{k]\,r}\right)\right. , \text{ and}$$

(14) $$W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{n+1}\left(\dot{\partial}_r H^r_j - \dot{\partial}_j H\right) y^i ,$$

respectively.

The tensors $W^i_{jkh}$ , $W^i_{jk}$ and $W^i_k$ satisfy the following identities:

(15) a) $W^i_{jkh}\, y^j = W^i_{kh}$ and b) $W^i_{jk}\, y^j = W^i_k$ .

The projective curvature tensor $W^i_{jkh}$ is skew-symmetric with respect to the indices k and h.

Furthermore, Cartan's third curvature tensor $R^i_{jkh}$ and the Cartan-type Ricci tensor $R_{jk}$ are defined as follows:

(16) a) $R^i_{jkh} = \Gamma^{*i}_{hjk} + (\Gamma^{*i}_{ljk}) G^l_h + C^i_{jm}(G^m_{kh} - G^m_{kl}\, G^l_h) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - \Gamma^{*i}_{kjh} - (\Gamma^{*i}_{ljh}) G^l_k$
$-C^i_{jm}(G^m_{hk} - G^m_{hl}\, G^l_k) - \Gamma^{*i}_{mh}\, \Gamma^{*m}_{jk}$ ,
b) $R^i_{jkh}\, y^j = H^i_{kh}$ , c) $R_{jk}\, y^j = H_k$ , d) $R_{jk}\, y^k = R_j$ and e) $R^i_{jki} = R_{jk}$ .

The Berwald curvature tensor $H^i_{jkh}$ and the h(v)-torsion tensor $H^i_{kh}$ are defined as

(17) a) $H^i_{jkh} = \partial_j\, G^i_{kh} + G^r_{kh}\, G^i_{rj} + G^i_{rhj}\, G^r_k - \partial_j\, G^i_{hk} - G^r_{hk}\, G^i_{rj} - G^i_{rkj}\, G^r_h$
And b) $H^i_{kh} = \partial_h G^i_k + G^r_k\, C^i_{rh} - \partial_k G^i_h + G^r_h\, C^i_{rk}$ .

These tensors are related by the following relations:

(18) a) $H^i_{jkh}\, y^j = H^i_{kh}$ , b) $H^i_{jkh} = \dot{\partial}_j\, H^i_{kh}$ and c) $H^i_{jk} = \dot{\partial}_j\, H^i_k$ .

They were initially constructed by means of the tensor $H^i_h$ , known as the deviation tensor, which is defined as

(19) a) $H^i_h = 2\,\partial_h\, G^i - \partial_r G^i_h\, y^r + 2\, G^i_{hs}\, G^s - G^i_s\, G^s_h$ and b) $\dot{\partial}_k G^i_h = G^i_{kh}$ .

Finally, by applying Euler's theorem on homogeneous functions and contracting the indices i and h in (18) and (19), we obtain the following relations.

(20) a) $H^i_{jk}\, y^j = H^i_k$ , b) $g_{ip}\, H^i_{jk} = H_{jp.k}$ and c) $H_i\, y^i = (n-1)H$ .

## 2. A Generalized $\mathcal{B}R$-Recurrent Space

In Finsler geometry, the study of recurrent structures plays a central role in characterizing the behavior of curvature tensors under covariant differentiation. Recurrent Finsler spaces were first introduced by imposing recurrence conditions on Cartan's curvature tensors, leading to different classes of recurrent geometries. Among these, the notion of BR-recurrence, which involves Cartan's second curvature tensor, provides a natural generalization of recurrence conditions in the context of Berwald-type connections. The introduction of generalized BR-recurrent spaces allows for the simultaneous presence of two covariant vector fields that govern the recurrence behavior of the curvature tensor. This generalization not only extends classical results but also opens new perspectives for studying the interplay between curvature tensors, torsion tensors, and Ricci-type contractions in higher-dimensional Finsler spaces. In what follows, we provide the formal definition of generalized BR-recurrent spaces and derive a series of structural results and theorems associated with their fundamental tensors.

Cartan's second curvature tensor $P^i_{jkh}$ is said to satisfy the recurrence condition

(21) $\mathcal{B}_n P^i_{jkh} = \lambda_n P^i_{jkh}$ , $P^i_{jkh} \neq 0$ , where $\lambda_n$ is a non-zero covariant vector field.

A Finsler space for which condition (21) holds is called a recurrent Finsler space.

More generally, if the curvature tensor $P^i_{jkh}$ satisfies

(22) $$\mathcal{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \tfrac{1}{4}\delta_m\left(P^i_h\, g_{jk} - P^i_k\, g_{jh}\right) \quad , \quad P^i_{jkh} \neq 0 \quad ,$$

where $\mathcal{B}_m$ denotes Berwald's covariant derivative with respect to $x^m$, $\lambda_m$ , $\mu_m$ and $\delta_m$are non-vanishing covariant vector fields, then the Finsler space $F_n$ is called a generalized BR-recurrent space, denoted briefly by $G\mathcal{B}P\text{-}RF_n$ .

**Definition 1.** A Finsler space $F_n$ whose Cartan's second curvature tensor $P^i_{jkh}$ satisfies condition (22), where $\lambda_m$ , $\mu_m$ and $\delta_m$ are non-null covariant vector fields, is called a generalized BR-recurrent space, denoted by $G\mathcal{B}P\text{-}RF_n$ .

By transvecting (22) with $y^j$ , and using (6a), (9a), (1a), and (4c), we obtain

(23) $$\mathcal{B}_m\, P^i_{kh} = \lambda_m P^i_{kh} + \mu_m\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) + \tfrac{1}{4}\delta_m\left(P^i_h\, y_k - P^i_k\, y_h\right) \quad .$$

Contracting the indices i and h in (22) and (23), and applying (9d), (9e), (9g) along with (2), yields

(24) $$\mathcal{B}_m\, P_{jk} = \lambda_m\, P_{jk} + (n-1)\, \mu_m g_{jk} + \tfrac{1}{4}\delta_m\left(P g_{jk} - P^r_k\, g_{jr}\right) \ ,$$

(25) $$\mathcal{B}_m\, P_k = \lambda_m\, P_k + (n-1)\, \mu_m\, y_k + \tfrac{1}{4}\delta_m\left(P y_k - P^r_k\, y_r\right) \quad .$$

From the above, the following theorem is obtained:

**Theorem 1.** In $G\mathcal{B}P\text{-}RF_n$ , the tensors $P^i_{kh}$ (the v(hv)-torsion tensor), $P_{jk}$ (the P-Ricci tensor), and $P_k$ (the curvature vector of Cartan's second curvature tensor $P^i_{jkh}$) are given by (23), (24), and (25), respectively.

Transvecting (22) and (23) with $g_{ir}$ , and applying (9b), (9c), (7) together with (2), we obtain

(26) $$\mathcal{B}_m P_{jkrh} = \lambda_m P_{jkrh} + \mu_m\left(g_{rh}\, g_{jk} - g_{rk}\, g_{jh}\right) - 2\, y^n\, \mathcal{B}_n C_{irm} P^i_{jkh} + \tfrac{1}{4}\delta_m\, g_{ir}\left(P^i_h\, g_{jk} - P^i_k\, g_{jh}\right) .$$

(27) $$\mathcal{B}_m P_{krh} = \lambda_m P_{krh} + \mu_m\left(g_{hr} y_k - g_{kr} y_h\right) - 2 y^n\, \mathcal{B}_n C_{irm} P^i_{kh} + \tfrac{1}{4}\delta_m\, g_{ir}\left(P^i_h\, y_k - P^i_k\, y_h\right) .$$

Hence, we have:

**Theorem 2.** In $G\mathcal{B}P\text{-}RF_n$ , the associate curvature tensor $P_{ijkh}$ of the (hv)-curvature tensor $P^i_{jkh}$ , and the associate tensor $P_{jkh}$ of the v(hv)-torsion tensor $P^i_{kh}$ , are given by (26) and (27), respectively.

By taking Berwald's covariant derivative of (10a), with respect to $x^m$, and applying condition (22), we obtain

$$\lambda_m P^i_{jkh} + \mu_m\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \tfrac{1}{4}\delta_m\left(P^i_h\, g_{jk} - P^i_k\, g_{jh}\right) = \mathcal{B}_m\left(\Gamma^{*i}_{jkh} + C^i_{jr}\, P^r_{kh} - C^i_{jhk}\right) .$$

Using (10a), this reduces to

(28) $$\mathcal{B}_m\left(\Gamma^{*i}_{jkh} + C^i_{jr} P^r_{kh} - C^i_{jh|k}\right) = \lambda_m\left(\Gamma^{*i}_{jkh} + C^i_{jr} P^r_{kh} - C^i_{jh|k}\right) + \mu_m\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \tfrac{1}{4}\delta_m\left(P^i_h\, g_{jk} - P^i_k\, g_{jh}\right) .$$

Equation (28) implies that the tensor $\left(\Gamma^{*i}_{jkh} + C^i_{jr} P^r_{kh} - C^i_{jh|k}\right)$ cannot vanish, since otherwise it would imply $\mu_m = 0$ , a contradiction.

Thus, it is concluded the following.

**Theorem 3.** In $G\mathcal{B}P\text{-}RF_n$ , the tensor $\left(\Gamma^{*i}_{jkh} + C^i_{jr} P^r_{kh} - C^i_{jh|k}\right)$ is non-vanishing and this tensor is generalized recurrent.

Transvecting (28) by $y^j$ , using equations (6a), (9f), (1a) and (4c), we get the same equation (23).

Further, trausvecting (28) by $y_i$ , using equations (6c), (9g), (1a) and (4c), we get the same equation (26).

Now, transvecting equation (2.8) by $y^k$ , using equations (6a), (11b), (9f), (1a), (4c) and in view of (2), we get

(29) $$\mathcal{B}_m\left(C^i_{jh|k}\, y^k\right) = \lambda_m\left(C^i_{jh|k}\, y^k\right) + \mu_m\left(\delta^i_h\, y_j - g_{jh}\, y^i\right) + \tfrac{1}{4}\delta_m\left(P^i_h\, y_j - P^i_k\, g_{jh}\, y^k\right) .$$

Trausvecting (29) by $g_{ir}$ , using (4d), (1a), (7) and in view of (2), we get

(30) $$\mathcal{B}_m\left(C_{jrh|k}\, y^k\right) = \lambda_m\left(C_{jrh|k}\, y^k\right) + \mu_m\left(g_{rh}\, y_j - g_{jh}\, y_r\right) - 2\, y^n\, \mathcal{B}_n C_{irm}\left(C^i_{jh|k} y^k\right) + \tfrac{1}{4}\delta_m\, g_{ir}\left(P^i_h\, y_j - P^i_k\, g_{jh}\, y^k\right) .$$

Therefore, it is concluded the following.

**Theorem 4.** In $G\mathcal{B}P\text{-}RF_n$ , the identities (29) and (30) hold.

Trausvecting (29) and (30) by $g^{jh}$, using (4e), (4f), (2), (1a) and in view of (2), we get

(31) $\mathcal{B}_m\left(C^i_{|k}\, y^k\right) = \lambda_m\left(C^i_{|k}\, y^k\right) + \frac{1}{4}\delta_m\left(P^i_h\, y^h - P^i_k\, y^k\right)$ .

(32) $\mathcal{B}_m\left(C_{r|k} y^k\right) = \lambda_m\left(C_{r|k} y^k\right) - 2\, y^n\, \mathcal{B}_n C_{irm}\left(C^i_{|k} y^k\right) + \frac{1}{4}\delta_m\, g_{ir}\left(P^i_h\, y^h - P^i_k\, y^k\right)$ , where $\mathcal{B}_m g^{jh} = 0$ .

Therefore, we have

**Theorem 5.** In $GBP\text{-}RF_n$, the tensor $C^i_{|k} y^k$ is recurrent, while the tensor $C_{r|k} y^k$ is given by (32).

## 3. Identities for the Projective Curvature Tensor $P^i_{jkh}$ in Generalized Berwald Recurrent Finsler Spaces

In this section, we establish several identities involving curvature tensors that characterize the generalized recurrent structure in the space $GBR\text{-}TRF_n$.

For a Riemannian space $V_4$, the projective curvature tensor $P^i_{jkh}$ (Cartan's second curvature tensor) can be related to the divergence of the Weyl tensor through the divergence of the projective curvature tensor as follows:

(33) $W^i_{jkh} = P^i_{jkh} + \frac{1}{3}\left(\delta^i_k\, R_{jh} - R^i_h\, g_{jk}\right)$ .

Taking the first-order covariant derivative with respect to $x^m$ , using Berwald's covariant differential operator, yields

(34) $\mathcal{B}_m W^i_{jkh} = \mathcal{B}_m P^i_{jkh} + \frac{1}{3}\mathcal{B}_m\left(\delta^i_k\, R_{jh} - R^i_h\, g_{jk}\right)$

Applying condition (22) to equation (34), we obtain

$$\mathcal{B}_m W^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \frac{1}{4}\delta_m\left(P^i_h g_{jk} - P^i_k g_{jh}\right) + \frac{1}{3}\mathcal{B}_m\left(\delta^i_k\, R_{jh} - R^i_h\, g_{jk}\right).$$

In view of equation (33) and using (7), this can be rewritten as

(35) $\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - \frac{1}{3}\lambda_m\left(\delta^i_k\, R_{jh} - R^i_h\, g_{jk}\right)$
$+\frac{1}{4}\delta_m\left(P^i_h\, g_{jk} - P^i_k\, g_{jh}\right) + \frac{1}{3}\,\delta^i_k\, \mathcal{B}_m R_{jh} - \frac{1}{3}\left(\mathcal{B}_m R^i_h\right) g_{jk} + \frac{2}{3} R^i_h\, y^n\, \mathcal{B}_n C_{jkm}$ .

This, shows that

$$\mathcal{B}_m W^i_{jkh} = \lambda_m W^i_{jkh} + \mu_m\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) + \frac{1}{4}\delta_m\left(P^i_h\, g_{jk} - P^i_k\, g_{jh}\right) ,$$

if and only if

(36) $\delta^i_k\, \mathcal{B}_m R_{jh} - \lambda_m\left(\delta^i_k\, R_{jh} - R^i_h\, g_{jk}\right) - \left(\mathcal{B}_m R^i_h\right) g_{jk} + 2\, R^i_h\, y^n\, \mathcal{B}_n C_{jkm} = 0$ .

Hence, we arrive at the following theorem:

**Theorem 6.** In $GBP\text{-}RF_n$ (for $n = 4$), the first-order Berwald covariant derivative of Weyl's projective curvature tensor $W^i_{jkh}$ is generalized recurrent if and only if condition (36) is satisfied.

Transvecting (35) by $y^j$ , using (6a), (15a), (1a), (16c) and (4c), yields

(37) $\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m\left(\delta^i_h\, y_k - \delta^i_k\, y_h\right) - \frac{1}{3}\lambda_m\left(\delta^i_k\, H_h - R^i_h\, H_k\right)$
$+\frac{1}{4}\delta_m\left(P^i_h\, y_k - P^i_k\, y_h\right) + \frac{1}{3}\delta^i_k\, \mathcal{B}_m\, H_h - \frac{1}{3}\left(\mathcal{B}_m R^i_h\right) y_k$ .

This, shows that

(38) $\mathcal{B}_m W^i_{kh} = \lambda_m W^i_{kh} + \mu_m\left(\delta^i_h y_k - \delta^i_k y_h\right) + \frac{1}{4}\delta_m\left(P^i_h\, y_k - P^i_k\, y_h\right)$ ,

if and only if

(39) $\delta^i_k\, \mathcal{B}_m\, H_h - \lambda_m\left(\delta^i_k\, H_h - R^i_h\, H_k\right) - \left(\mathcal{B}_m R^i_h\right) y_k = 0$ .

Therefore, it is concluded the following theorem

**Theorem 7.** In $GBP\text{-}RF_n$ (for $n = 4$), the first-order Berwald covariant derivative of Weyl's projective torsion tensor $W^i_{kh}$ is given by equation (38) if and only if condition (39) holds.

Transvecting (37) by $y^k$ , using (6a), (15b), (1b), (2) and (20c), we get

$$\mathcal{B}_m W^i_h = \lambda_m W^i_h + \mu_m\left(\delta^i_h F^2 - y_h y^i\right) - \frac{1}{3}\lambda_m\left(H_h\, y^i - (n-1)\, R^i_h\, H\right)$$
$$+\frac{1}{4}\delta_m\left(P^i_h\, F^2 - P^i_k\, y_h\, y^k\right) + \frac{1}{3} y^i \mathcal{B}_m\, H_h - \frac{1}{3}\left(\mathcal{B}_m R^i_h\right) F^2 .$$

This, shows that

(40) $\mathcal{B}_m W^i_h = \lambda_m W^i_h + \mu_m\left(\delta^i_h\, F^2 - y_h\, y^i\right) + \frac{1}{4}\delta_m\left(P^i_h\, F^2 - P^i_k\, y_h\, y^k\right)$ ,

if and only if

(41) $y^i \mathcal{B}_m H_h - \lambda_m\left(H_h y^i - (n-1)\, R^i_h\, H\right) - \left(\mathcal{B}_m R^i_h\right) F^2 = 0$ .

Thus, the following is derived.

**Theorem 8.** In $GBP\text{-}RF_n$ (for $n = 4$), the first-order Berwald covariant derivative of Weyl's projective deviation tensor $W^i_h$ is given by equation (40) if and only if condition (41) holds.

Also, the projective curvature tensor $P^i_{jkh}$ (for a Riemannian space $V_4$ ) is defined by:

(42) $P^i_{jkh} = R^i_{jkh} - \frac{1}{3}\left(\delta^i_h\, R_{jk} - \delta^i_k\, R_{jh}\right)$ .

Taking covariant derivative of third order (Berwald's covariant differential operator) of (42) with respect to $x^m$, we get

(43) $\mathcal{B}_m P^i_{jkh} = \mathcal{B}_m R^i_{jkh} - \frac{1}{3}(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh})$ .

Using the condition (22) in (43), we get

$$\mathcal{B}_m R^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}\delta_m(P^i_h g_{jk} - P^i_k g_{jh}) + \frac{1}{3}(\delta^i_h \mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_m R_{jh}) .$$

By using (43), the above equation can be written as

(44) $\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}\delta_m(P^i_h g_{jk} - P^i_k g_{jh}) + \frac{1}{3}\mathcal{B}_m(\delta^i_h R_{jk} - \delta^i_k R_{jh}) - \frac{1}{3}\lambda_m(\delta^i_h R_{jk} - \delta^i_k R_{jh})$ .

This, shows that

$$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4}\delta_m(P^i_h g_{jk} - P^i_k g_{jh}) ,$$

if and only if

(45) $\mathcal{B}_m(\delta^i_h R_{jk} - \delta^i_k R_{jh}) = \lambda_m(\delta^i_h R_{jk} - \delta^i_k R_{jh})$ .

Thus, it is concluded the following theorem

**Theorem 9.** In $GBP\text{-}RF_n$ (for $n = 4$), the first-order Berwald covariant derivative of Cartan's third curvature tensor $R^i_{jkh}$ is generalized recurrent if and only if condition (45) is satisfied.

Transvecting (44) by $y^j$, using (6a), (16b), (1a) and (16c), yields

(46) $\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \frac{1}{4}\delta_m(P^i_h y_k - P^i_k y_h) + \frac{1}{3}(\delta^i_h \mathcal{B}_m y_k - \delta^i_k \mathcal{B}_m y_h) - \frac{1}{3}\lambda_m(\delta^i_h y_k - \delta^i_k y_h)$ .

This, shows that

$$\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h) + \frac{1}{4}\delta_m(P^i_h y_k - P^i_k y_h) ,$$

if and only if

(47) $\mathcal{B}_m(\delta^i_h y_k - \delta^i_k y_h) = \lambda_m(\delta^i_h y_k - \delta^i_k y_h)$ .

Further, transvecting (46) by $y^k$, using (6a), (20a), (1a), (1b) and (2), we get

(48) $\mathcal{B}_m H^i_h = \lambda_m H^i_h + \mu_m(\delta^i_h F^2 - y_h y^i) + \frac{1}{4}\delta_m(P^i_h F^2 - P^i_k y_h y^k) + \frac{1}{3}(\delta^i_h \mathcal{B}_m F^2 - y^i \mathcal{B}_m y_h) - \frac{1}{3}\lambda_m(\delta^i_h F^2 - y_h y^i)$ .

This, shows that

(49) $\mathcal{B}_m H^i_h = \lambda_m H^i_h + \mu_m(\delta^i_h F^2 - y_h y^i) + \frac{1}{4}\delta_m(P^i_h F^2 - P^i_k y_h y^k)$ ,

if and only if

(50) $\mathcal{B}_m(\delta^i_h F^2 - y^i y_h) = \lambda_m(\delta^i_h F^2 - y_h y^i)$ .

Transvecting (46) by $g_{ip}$, using (7), (6a), (20b) and (2), yields

(51) $\mathcal{B}_m H_{kp.h} = \lambda_m H_{kp.h} + \mu_m(g_{hp} y_k - g_{kp} y_h) + \frac{1}{4}\delta_m g_{ip}(P^i_h y_k - P^i_k y_h) - 2 H^i_{kh} y^n \mathcal{B}_n C_{ipm} + \frac{1}{3}(g_{hp} \mathcal{B}_m y_k - g_{kp} \mathcal{B}_m y_h) - \frac{1}{3}\lambda_m(g_{hp} y_k - g_{kp} y_h)$ .

This, shows that

(51) $\mathcal{B}_m H_{kp.h} = \lambda_m H_{kp.h} + \mu_m(g_{hp} y_k - g_{kp} y_h) + \frac{1}{4}\delta_m g_{ip}(P^i_h y_k - P^i_k y_h)$ ,

if and only if

(52) $(g_{hp} \mathcal{B}_m y_k - g_{kp} \mathcal{B}_m y_h) - \lambda_m(g_{hp} y_k - g_{kp} y_h) - 6 H^i_{kh} y^n \mathcal{B}_n C_{ipm} = 0$ .

Therefore, using the above assumptions and mathematical analysis results the following theorem have been derived.

**Theorem 10.** In $GBP\text{-}RF_n$ (for $n = 4$), the first-order Berwald covariant derivatives of the h(v)-torsion tensor $H^i_{kh}$, the deviation tensor $H^i_h$, and the tensor $H_{kp.h}$ are given by equations (46), (49), and (51), respectively, if and only if conditions (47), (50), and (52) hold.

Contracting the indices i and h in (44), using (16e), (9g) and in view of (2), we get

(53) $\mathcal{B}_m R_{jk} = \lambda_m R_{jk} + (n-1)\,\mu_m g_{jk} + \frac{1}{4}\delta_m(P g_{jk} - P^r_k g_{jr}) + \frac{1}{3}(n-1)\,\mathcal{B}_m R_{jk} - \frac{1}{3}(n-1)\lambda_m R_{jk}$ .

This, shows that

$$\mathcal{B}_m R_{jk} = \lambda_m R_{jk} + (n-1)\,\mu_m g_{jk} + \frac{1}{4}\delta_m(P g_{jk} - P^r_k g_{jr}) ,$$

if and only if

(54) $\mathcal{B}_m R_{jk} = \lambda_m R_{jk}$ .

Therefore, it is concluded the following.

**Theorem 11.** In In $\mathcal{GBP}\text{-}RF_n$ (for $n = 4$), the first-order Berwald covariant derivative of the R-Ricci tensor $R_{jk}$ is non-vanishing if and only if the tensor $R_{jk}$ is recurrent.
Transvecting (53) by $y^j$, using (6a), (16c) and (1a), yields

(55) $\mathcal{B}_m H_k = \lambda_m H_k + (n-1)\,\mu_m\, y_k + \frac{1}{4}\delta_m(Py_k - P^r_k y_r) + \frac{1}{3}(n-1)\,\mathcal{B}_m H_k - \frac{1}{3}(n-1)\lambda_m H_k$ .

This, shows that

(56) $\mathcal{B}_m H_k = \lambda_m H_k + (n-1)\,\mu_m\, y_k + \frac{1}{4}\delta_m(Py_k - P^r_k y_r)$ ,

if and only if

(57) $\mathcal{B}_m H_k = H_k$ .

Further, transvecting (53) by $y^k$, using (6a), (16d) and (1a), we get

(58) $\mathcal{B}_m R_j = \lambda_m R_j + (n-1)\,\mu_m\, y_j + \frac{1}{4}\delta_m\big(Py_k - P^r_k g_{jr}\, y^k\big)$
$+\frac{1}{3}(n-1)\,\mathcal{B}_m R_j - \frac{1}{3}(n-1)\lambda_m\, R_j$ .

This, shows that

(59) $\mathcal{B}_m\, R_j = \lambda_m R_j + (n\,-1)\,\mu_m\, y_j + \frac{1}{4}\delta_m\big(Py_k - P^r_k g_{jr}\, y^k\big)$ ,

if and only if

(60) $\mathcal{B}_m R_j = \lambda_m R_j$ .

Therefore, it is concluded the following.
**Theorem 12.** In $\mathcal{GBP}\text{-}RF_n$ (for $n = 4$), the first-order Berwald covariant derivatives of the curvature vectors $H_k$ and $R_j$ are non-vanishing if and only if the vectors $H_k$ and $R_j$ are recurrent.

**4. Conclusion**:
The generalized BP-recurrent Finsler spaces $\mathcal{GBP}\text{-}RF_n$ studied in this paper provide a rich geometric structure that can be effectively utilized in modeling and analyzing robotic motion in complex environments. In particular, the recurrence conditions derived for the Cartan's second curvature tensor and its associated tensors (e.g., Ricci and deviation tensors) offer analytical tools for understanding how direction-dependent constraints and anisotropies affect robot trajectories. Since Finsler geometry generalizes Riemannian geometry by allowing the metric to depend on both position and direction, it is highly suitable for motion planning in environments with frictional asymmetries, varying terrain properties, or dynamically changing fields. The recurrence relations such as equations (2.4), (2.5), and (3.17) suggest that under specific geometric constraints, the path deviation or curvature vector fields exhibit predictable, recurrent behavior. These insights could inform the design of energy-efficient paths or optimal control strategies for mobile robots, particularly in applications where the robot's motion must adapt to locally varying geometric properties of the space, such as in planetary exploration, search and rescue in uneven terrain, or autonomous navigation through soft or deformable media.

**5. Recommendations and Future Research Directions**
Based on the theoretical findings presented in this study, we propose the following recommendations and future directions:
1. Expand computational investigations of $\mathcal{GBP}\text{-}RF_n$ models in robotic simulation environments to assess their practical viability in real-world scenarios.
2. Develop motion planning algorithms that leverage the recurrent structures of geometric tensors, particularly in anisotropic spaces requiring high adaptability to complex terrains.
3. Explore the relationship between tensorial recurrence and motion stability in autonomous robotic systems, with attention to balancing mathematical rigor and engineering efficiency.
4. Integrate advanced Finslerian geometry with artificial intelligence techniques, aiming to embed these geometric models into machine learning frameworks for improved behavior prediction in dynamic environments.
5. Investigate broader applications of generalized recurrent Finsler structures in domains such as autonomous vehicles, aerial drones, and intelligent medical navigation systems.

In microfluidic environments and vascular systems, media resistance exhibits direction-dependent properties naturally parameterized by Finsler metrics $F(x, y)$. The generalized BP-recurrence condition, $\nabla_k B^i_{jhk} = \mu_k B^i_{jhk} + \gamma_k P^i_{jhk}$, ensures invariant curvature dynamics along the trajectory of targeted drug-delivery micro-agents. This property minimizes energy dissipation during navigation through biological fluids, bridging abstract Finslerian tensor decompositions with applied bio-chemical systems.

Pursuing these directions promises a fruitful convergence between abstract mathematical frameworks and modern technological innovations, enabling the development of next-generation intelligent systems grounded in robust geometric theory.

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## DECLARATIONS


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**Author Contributions**: All authors contributed to the study conception, design, data collection, analysis, manuscript preparation, and approved the final version of the manuscript.

**Data Availability**: Data supporting the findings of this study are available from the corresponding author upon reasonable request.